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Matrices for Affine Hecke Modules

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INTRODUCTION

The representation theory of a reductive p-adic group G seems to be locally modeled on the representation theory of affine Hecke algebras or closely related algebras. More precisely, one can often find subcategories of the category of admissible representations of G, which are Morita equivalent to module categories over various affine Hecke algebras, and the hope is that all admissible representations may be thus described. See, for example, [BK] and [HM] for GL_n, [Ki] for other classical groups, [L1] for unipotent and [M] for level-zero representations, and [Ro] for the ramified principal series. The advantages of this approach depend in part on the possibility of explicit calculations in affine Hecke modules.

To describe the calculations we have in mind, recall that an affine Hecke algebra $\mathscr{H}$ is generated by two subalgebras, $\mathscr{H}_0$ and $\mathscr{A}$, where $\mathscr{A}$ is the coordinate ring of a complex torus $\mathbf{T}$, which is a maximal torus in a reductive Lie group $\mathbf{G}$, and $\mathscr{H}_0$ is generated by operators T_s, where s runs over a fixed set Σ of simple reflections in the Weyl group W of $\mathbf{G}$, satisfying the usual braid relations along with the relation $(T_s - q_s)(T_s + 1) = 0$ for certain parameters $q_s > 0$ (see Section 1 for more details). For reasons that will become clear, we are especially interested in the case of "unequal parameters," that is, when the map $s \mapsto q_s$ is not constant.

Let E be a finite-dimensional $\mathscr{H}$-module. Its restriction to the commutative algebra $\mathscr{A}$ decomposes as

$$E = \bigoplus_{\tau \in \mathbf{T}} E_\tau,$$

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where the "weight space" E_τ consists of the vectors in E annihilated by some power of the maximal ideal $\mathfrak{m}_\tau$ in $\mathscr{A}$. The essential problem is to calculate the E_τ, at least as vector spaces and ultimately as $\mathscr{A}$-modules. For a wide class of simple $\mathscr{H}$-modules E, we give here an explicit algebro-geometric description of each weight space E_τ. Our result is new for Hecke algebras with unequal parameters and is also new when E does not contain the trivial or sign characters of $\mathscr{H}_0$. We also describe the action of T_s on each sum of pairs $E_\tau \oplus E_{s\tau}$ (the sum is preserved by T_s), which in principle gives the complete structure of the $\mathscr{H}$-module E.

When E comes from a representation V of G by some categorical equivalences as above, then the actions of $\mathscr{A}$ and $\mathscr{H}_0$ determine the irreducibility of induced representations, square-integrability/temperedness, restriction to maximal compact subgroups, and other essential features of an admissible representation. Some of these deductions are illustrated in Section 7 below for a particular square-integrable unipotent representation of E_8 arising in [R4]. The detailed treatment of this example was deferred to the present paper as it requires the results given herein.

Of course, we need some information about the $\mathscr{H}$-module E to begin such computations. Every simple module E can be embedded in a principal series module M, and we suppose that the embedding of one weight space $E_\tau \subseteq M_\tau$ is known. Then, in principle, it suffices to describe the actions of $\mathscr{A}$ and T_s on M.

In [R1] we constructed an explicit basis for each weight space M_τ in M. Here, in Section 2, we give the matrices for $\mathscr{A}$ and T_s on M in terms of this basis. Actually, we work with certain operators F_s which map E_τ to $E_{s\tau}$, from which the action of T_s is easily recovered.

We have twice hedged with "in principle," inviting the suspicion that our explicit formulas may contain practical difficulties. For arbitrary simple $\mathscr{H}$-modules E this is so. The entries in our matrices share many properties with Kazhdan–Lusztig polynomials (see Section 6) and, in particular, are only defined recursively.

However, for certain simple $\mathscr{H}$-modules E we can refine these to give effective formulas for the weight spaces E_τ, in terms of partial derivatives (cap-products, in geometric terms).

These special E's are those containing the following kind of weight. Say that $\tau \in \mathbf{T}$ has "standard singularity of type J" if the centralizer in $\mathbf{G}$ of τ is connected and if the centralizer in W of the hyperbolic part τ_h of τ is the subgroup W_J generated by a set of simple reflections $J \subseteq \Sigma$. If τ has standard singularity, then there is a unique simple $\mathscr{H}$-module E with $E_\tau \neq 0$, and we say that E has standard singularity as well. If $\mathbf{G}$ is simply connected, then each principal series M (equivalently, each category of $\mathscr{H}$-modules with given central character) contains at least one irreducible constituent E with standard singularity.

The main result in this paper is an explicit formula for the weight spaces in a simple $\mathcal{H}$-module with standard singularity. It is valid for any parameter set $\{q_s\}$.

Suppose $\tau \in \mathbf{T}$ has standard singularity of type J. Let W^J be the set of shortest representatives for W/W_J. Every $w \in W$ may be uniquely expressed as $w = yz$ with $y \in W^J$, $z \in W_J$. For any $x \in W$, let $\mathcal{B}_{x\tau}$ be the flag variety of the centralizer of $x\tau$ in $\mathbf{G}$. Let H^* and H_* denote singular cohomology and homology with complex coefficients. There is a natural surjective ring homomorphism,

$$j_{x\tau}\colon \mathcal{A} \longrightarrow H^*(\mathcal{B}_{x\tau}),$$

by which $H_*(\mathcal{B}_{x\tau})$ becomes an $\mathcal{A}$-module under the cap-product. Now we can state our main result.

THEOREM. *Suppose τ has standard singularity of type J and that E is the unique simple $\mathcal{H}$-module with $E_\tau \neq 0$. Let $w = yz$, as above.*

(1) *As $\mathcal{A}$-modules, we have $M_{w\tau} \simeq H_*(\mathcal{B}_{w\tau})$, which in turn is isomorphic to the twist by y of the $\mathcal{A}$-module $H_*(\mathcal{B}_{z\tau})$.*

(2) *The $\mathcal{A}$-module $E_{w\tau}$ is isomorphic to the twist by y of the $\mathcal{A}$-submodule of $H_*(\mathcal{B}_{z\tau})$ generated by the cap-product*

$$j_{z\tau}\Bigg(\prod_{\beta\in R_{z\tau}(y)} e_\beta - e_\beta(z\tau)\Bigg) \cap [\mathcal{B}_{z\tau}],$$

where $R_{z\tau}(y)$ is a certain set of roots depending on y, $z\tau$, and the parameters defining $\mathcal{H}$ (see (5.5c)), and $[\mathcal{B}_{z\tau}]$ is the fundamental class of $\mathcal{B}_{z\tau}$.

(3) *The dimension of $E_{w\tau}$ equals that of the span of all partial derivatives of the polynomial*

$$\Bigg(\prod_{\beta\in R_{z\tau}(y)} \partial_\beta\Bigg)\Pi_{z\tau},$$

where $\Pi_{z\tau}$ is the harmonic polynomial corresponding to $[\mathcal{B}_{z\tau}]$ and ∂_β is the derivation on polynomials extending β.

The proof is given in Section 5. It is item (3) that solves our computational problem effectively for simple $\mathcal{H}$-modules with standard singularity.

The theorem was already known for certain modules. If J has one element, it was proved in [R1, (10.11)]. If $\mathcal{H}$ has equal parameters $q_s \equiv q$, then the simple $\mathcal{H}$-modules containing the trivial and sign characters of $\mathcal{H}_0$ have standard singularity. For these modules, the theorem was proved in [R2, Section 5] using Whittaker functions and can be deduced from the geometric view of Hecke algebras in [KL] (see also [CG]). Moreover, if E contains the sign character of $\mathcal{H}_0$, then the generator in (3) is a scalar times the harmonic polynomial corresponding to the fundamental class of a connected component of the variety attached to E in [KL].

For unequal parameters, our theorem is incomplete on this last point, since it gives no interpretation of the polynomial in (3) as a geometrically defined cycle. However, as evidence for a larger geometric picture in the unequal parameter case, we prove en route (see Section 4) the following.

PROPOSITION. *Let* $\tau \in \mathbf{T}$, *and let* M *be a principal series module with* $M_\tau \neq 0$. *Then*

(1) *If* E *is any subquotient of* M, *then the* $\mathcal{A}$ *action on* E_τ *factors through* j_τ, *so that* E_τ *is an* $H^*(\mathcal{B}_\tau)$*-module.*

(2) *The* $\mathcal{A}$*-module* M_τ *is isomorphic to* $H_*(\mathcal{B}_\tau)$ *if and only if it is cyclic.*

The final section, 7, contains the promised application to unipotent representations of E_8.

1. LOCALIZED HECKE ALGEBRAS

For more details in this section, see [R1, Sections 1–6]. The main new result here is (1.9) below, which describes multiplication in a localized Hecke algebra (which is no longer an algebra), which will lead to our principal series matrices.

(1.1) We begin with a complex reductive Lie group $\mathbf{G}$, with maximal torus $\mathbf{T}$, having roots, positive roots, and simple roots Δ, Δ^+, and Σ, respectively, and Weyl group W. We assume that this root system is irreducible. For $w \in W$, let $l(w)$ be the length of w, and let $N(w)$ be the set of positive roots made negative by w. The W-action on $\mathbf{T}$ is denoted $(w, \tau) \mapsto w\tau$, and W_τ is the stabilizer in W of $\tau \in \mathbf{T}$.

Let $\mathcal{A} = \mathbb{C}[\mathbf{T}]$ be the ring of regular functions on $\mathbf{T}$, and let $\mathcal{K} = \mathbb{C}(\mathbf{T})$ be the field of rational functions on $\mathbf{T}$. The Weyl group acts on $\mathcal{A}$ and $\mathcal{K}$ by $f^w(\tau) = f(w\tau)$. Let $X^*(\mathbf{T})$ be the character lattice of $\mathbf{T}$. For $\lambda \in X^*(\mathbf{T})$, we write $e_\lambda \in \mathcal{A}$ for the corresponding character. Then $e_\lambda^w = e_{w^{-1}\lambda}$.

For $\tau \in \mathbf{T}$, let $\mathfrak{m}_\tau$ be the maximal ideal of $\mathcal{A}$ at τ, and let $\mathcal{A}_\tau \subset \mathcal{K}$ be the localization of $\mathcal{A}$ at $\mathfrak{m}_\tau$. So $\mathcal{A}_\tau$ consists of those rational functions which are holomorphic at τ, and $\widetilde{\mathfrak{m}}_\tau := \mathcal{A}_\tau \mathfrak{m}_\tau$ is the maximal ideal of $\mathcal{A}_\tau$.

(1.2) In this paper, an affine Hecke algebra $\mathcal{H}$ attached to $\mathbf{G}$ is defined by a collection of positive real numbers

$$\{q_0, q_\beta \colon \beta \in \Delta\},$$

with $q_{w\beta} = q_\beta$ for all $w \in W$, as follows.

First let $\mathscr{H}_0$ be the Hecke algebra of W, with parameters $\{q_\beta\}$. It has a $\mathbb{C}$-basis $\{T_w : w \in W\}$ with multiplication rules $T_x T_y = T_{xy}$ if $l(xy) = l(x) + l(y)$, and $(T_{s_\alpha} - q_\alpha)(T_{s_\alpha} + 1) = 0$, for a simple root $\alpha \in \Sigma$. Let us write

$$B_{s_\alpha} = T_{s_\alpha} - q_\alpha, \qquad \alpha \in \Sigma.$$

Next, for each $\beta \in \Delta$ define the rational function $\zeta_\beta \in \mathscr{K}$, as follows. If $\mathbf{G} = SO_{2n+1}$ and β is a short root, then

$$\zeta_\beta = \frac{(q_\beta^{1/2} q_0^{1/2} - e_\beta)(q_\beta^{1/2} q_0^{-1/2} + e_\beta)}{1 - e_{2\beta}}. \tag{1.2a}$$

Via the symmetry of the corresponding affine Dynkin diagram (of type $\widetilde{C}_n$), we may assume $q_0 \leq q_\beta$.

In all other cases,

$$\zeta_\beta = \frac{q_\beta - e_\beta}{1 - e_\beta}. \tag{1.2b}$$

Then the affine Hecke algebra $\mathscr{H}$ is a twisted tensor product of two subalgebras

$$\mathscr{H} = \mathscr{H}_0 \tilde{\otimes}_{\mathbb{C}} \mathscr{A},$$

where the cross multiplication is given, for a simple reflection $s = s_\alpha$, by

$$\theta B_s = B_s \theta^s + (\theta^s - \theta)\zeta_\alpha, \quad \theta \in \mathscr{A}. \tag{1.2c}$$

(1.3) Let $\tau \in \mathbf{T}$. Corresponding to $\mathscr{A} \subset \mathscr{A}_\tau \subset \mathscr{K}$, we have $\mathscr{H} \subset \mathscr{H}_\tau \subset \mathscr{H}_{\mathscr{K}}$, where

$$\mathscr{H}_\tau = \mathscr{H} \otimes_{\mathscr{A}} \mathscr{A}_\tau = \mathscr{H}_0 \otimes_{\mathbb{C}} \mathscr{A}_\tau, \qquad \mathscr{H}_{\mathscr{K}} = \mathscr{H} \otimes_{\mathscr{A}} \mathscr{K} = \mathscr{H}_0 \tilde{\otimes}_{\mathbb{C}} \mathscr{K}.$$

Note that $\mathscr{H}_{\mathscr{K}}$ is an algebra over the W-invariants $\mathscr{K}^W$, not over $\mathscr{K}$, and $\mathscr{H}_\tau$ is not an algebra in general, only an $\mathscr{H} - \mathscr{A}_\tau$ bimodule.

We have an evaluation homomorphism

$$F \mapsto F(\tau) \colon \mathscr{H}_\tau \longrightarrow \mathscr{H}_0,$$

given on pure tensors by $T \otimes \theta \mapsto \theta(\tau) T$, for $T \in \mathscr{H}_0,\ \theta \in \mathscr{A}_\tau$.

Let

$$\mathbb{C}_\tau = \mathscr{A}/\mathfrak{m}_\tau = \mathscr{A}_\tau/\widetilde{\mathfrak{m}}_\tau,$$

identified with $\mathbb{C}$ via evaluation at τ. Define the principal series $\mathscr{H}$-module

$$M(\tau) = \mathscr{H} \otimes_{\mathscr{A}} \mathbb{C}_\tau = \mathscr{H}_\tau \otimes_{\mathscr{A}_\tau} \mathbb{C}_\tau.$$

The vector $v_\tau := 1 \otimes 1 \in M(\tau)$ generates $M(\tau)$ over $\mathscr{H}_0$.

(1.4) For each simple reflection $s = s_\alpha$, let

$$F_s = B_s + \zeta_\alpha \in \mathcal{H}_{\mathcal{K}}.$$

If $w = s_k \cdots s_1$ is a reduced expression, we let

$$F_w = F_{s_r} \cdots F_{s_1},$$

the product taken in $\mathcal{H}_{\mathcal{K}}$. By [R1, (4.3)], F_w is independent of the reduced expression chosen for w and $\{F_w : w \in W\}$ is a (left and right) $\mathcal{K}$-basis of $\mathcal{H}_{\mathcal{K}}$. By (1.2c) we have

$$\theta F_w = F_w \theta^w, \qquad \theta \in \mathcal{K}, \tag{1.4a}$$

which, along with [R1, (4.3)(2)], implies that

$$F_x F_y = F_{xy} \eta_{x,y}, \tag{1.4b}$$

where

$$\eta_{x,y} = \prod_{\substack{\beta \in N(y) \\ xy\beta > 0}} \zeta_\beta \zeta_{-\beta}.$$

In particular, we have

$$F_s F_y = F_{sy} \eta_{s,y},$$

where

$$\eta_{s,y} = \begin{cases} [\zeta_\alpha \zeta_{-\alpha}]^y & \text{if } sy < y \\ 1 & \text{if } sy > y. \end{cases}$$

If $sx > x$, then computing $F_s F_x F_y$ in two ways using (1.4b) shows that

$$\eta_{s,xy} \eta_{x,y} = \eta_{sx,y} \qquad (sx > x). \tag{1.4c}$$

(1.5) Let $\tau_0 \in \mathbf{T}$ be a common zero of all ζ_α, for $\alpha \in \Sigma$. All F_w belong to $\mathcal{H}_{\tau_0}$, and we define

$$B_w = F_w(\tau_0).$$

Then $\{B_w : w \in W\}$ is a new basis of $\mathcal{H}_0$, which we use from now on [R1, Section 5]. From (1.4a), (1.4b) we get the multiplication rule

$$B_s B_w = \eta_{s,w}(\tau_0) B_{sw} - \zeta_\alpha^w(\tau_0) B_w \tag{1.5a}$$

(1.6) Define rational functions $p_{x,y} \in \mathcal{K}$ by the expansion

$$F_y = \sum_x B_x p_{x,y}.$$

Then $p_{x,x} = 1$ and $p_{x,y} \neq 0 \Rightarrow x \leq y$, under the Bruhat order of W. From (1.5a) we get the following recursive formula for $p_{x,y}$:

$$p_{x,sw} = [\zeta_\alpha^w - \zeta_\alpha^x(\tau_0)]p_{x,w} + \eta_{s,sx}(\tau_0)p_{sx,w} \qquad \text{for } w < sw. \tag{1.6a}$$

I do not know a closed form for $p_{x,y}$, except in particular cases. For example [R1, Section 5], we have

$$p_{e,w} = \prod_{\beta \in N(w)} \zeta_\beta. \tag{1.6b}$$

We also have

LEMMA 1.7. *Suppose $x < y$ are adjacent in the Bruhat order, so $l(y) = l(x) + 1$ and there is a positive root β with $y = xs_\beta$. Then*

$$p_{x,y} = \zeta_\beta - \zeta_\beta(\tau_0).$$

Proof. There are $u, v \in W$ and a simple reflection $s = s_\alpha$, such that $x = uv$, $y = usv$, with additive lengths in these expressions. Note that $v\beta = \alpha$. We assume u to be chosen to have minimal length. If $u = 1$, our assertion is immediate from (1.6a). If $u \neq 1$, suppose $t = s_\gamma$ is a simple reflection such that $tu < u$. By induction, we have

$$p_{tx,ty} = \zeta_\beta - \zeta_\beta(\tau_0),$$

so we have to show that $p_{x,y} = p_{tx,ty}$. By (1.6a) it suffices to show that $x \not\leq ty$. Since both elements have the same length, it suffices to show that $x \neq ty$. But $x = ty$ would contradict the minimality of u. ∎

For additional properties of $p_{x,y}$, see Section 6.

(1.8) Let $\tau \in \mathbf{T}$ and $w \in W$. Let $\mathcal{H}_{\mathcal{K}}^{w\tau}$ denote the $\mathcal{K}$-span (left or right, it is the same) of $\{F_x : x\tau = w\tau\}$. Define

$$\mathcal{H}_{w,\tau} = \mathcal{H}_\tau \cap \mathcal{H}_{\mathcal{K}}^{w\tau}.$$

Then [R1, Section 6] we have a decomposition

$$\mathcal{H}_\tau = \bigoplus_{w \in W/W_\tau} \mathcal{H}_{w,\tau}, \tag{1.8a}$$

and $\mathcal{H}_{w,\tau}$ is a free right $\mathcal{A}_\tau$-module. An $\mathcal{A}_\tau$-basis $\{H_x : x\tau = w\tau\}$ of $\mathcal{H}_{w,\tau}$ is constructed as follows. By (1.6), there are unique rational functions $r_{x,y} \in \mathcal{K}$ such that

$$\sum_{y\tau = w\tau} p_{z,y} r_{y,x} = \delta_{z,x},$$

where $\delta_{z,x} = 1$ if $z = x$, $\delta_{z,x} = 0$ if $z \neq x$. Then the basis elements H_x are given by

$$H_x = \sum_{y\tau = w\tau} F_y r_{y,x}. \tag{1.8b}$$

We also have

$$F_x = \sum_{y\tau = w\tau} H_y p_{y,x}. \tag{1.8c}$$

Finally, the expansion of H_x in terms of B_y's has the form

$$H_x = B_x + \sum_{y\tau \neq w\tau} B_y h_{y,x}, \qquad h_{y,x} \in \mathscr{A}_\tau. \tag{1.8d}$$

PROPOSITION 1.9. *The left multiplication of $\mathscr{H}$ on $\mathscr{H}_{w,\tau}$ is given in terms of the basis $\{H_x : x\tau = w\tau\}$ as follows*:

(1) *We have $\mathscr{A}_{w\tau}\mathscr{H}_{w,\tau} = \mathscr{H}_{w,\tau}$, with multiplication formula*

$$\theta H_x = \sum_{z\tau = w\tau} H_z \Bigg[\sum_{y\tau = w\tau} p_{z,y} \theta^y r_{y,x} \Bigg], \qquad \theta \in \mathscr{A}_{w\tau}.$$

In particular, the term in brackets belongs to $\mathscr{A}_\tau$.

(2) *We have $\widetilde{\mathfrak{m}}_{w\tau}^n \mathscr{H}_{w,\tau} \subseteq \mathscr{H}_{w,\tau}\widetilde{\mathfrak{m}}_\tau$, where $n = |W_\tau|$.*

(3) *If $s = s_\alpha$ is a simple reflection such that $sw\tau \neq w\tau$, then $F_s\mathscr{H}_{w,\tau} \subseteq \mathscr{H}_{sw,\tau}$, with multiplication formula*

$$F_s H_x = \sum_{z\tau = w\tau} H_{sz} \Bigg[\sum_{y\tau = w\tau} p_{sz,sy} \eta_{s,y} r_{y,x} \Bigg].$$

(4) *If $sw\tau = w\tau$, then $B_s\mathscr{H}_{w,\tau} \subseteq \mathscr{H}_{w,\tau}$, with multiplication formula*

$$B_s H_x = \eta_{s,x}(\tau_0) H_{sx} - \zeta_\alpha^x(\tau_0) H_x.$$

Proof. We have $\mathscr{A}\mathscr{H}_\tau = \mathscr{H}_\tau$ by definition, and $\mathscr{A}\mathscr{H}_{\mathscr{K}}^{w\tau} = \mathscr{H}_{\mathscr{K}}^{w\tau}$ by (1.4a), so at least $\mathscr{A}\mathscr{H}_{w,\tau} = \mathscr{H}_{w,\tau}$. For any $\theta \in \mathscr{K}$, the formula in (1.9)(1) follows from relation (1.4a). By (1.8b), the coefficient

$$\sum_{y\tau = w\tau} p_{z,y} \theta^y r_{y,x} \tag{1.9a}$$

in (1.9)(1) is also the coefficient of B_z in θH_x, hence it belongs to $\mathscr{A}_\tau$, at least if $\theta \in \mathscr{A}$.

If, moreover, $\theta \in \mathscr{A}$ is holomorphic and nonzero at $w\tau$, then the coefficients (1.9a) form an upper triangular matrix (for an appropriate ordering on wW_τ) whose diagonal entries are units in $\mathcal{A}_\tau$ and whose entries above the diagonal are in $\mathcal{A}_\tau$. Hence the inverse matrix has entries in $\mathcal{A}_\tau$, so

$$\sum_{z\tau=w\tau} H_z\left[\sum_{y\tau=w\tau} p_{z,y}(\theta^{-1})^y r_{y,x}\right] \in \mathcal{H}_{w,\tau},$$

proving (1).

By [R1, (6.8)], the vectors $H_x v_\tau$, for $x\tau = w\tau$, form a basis of the space $M(\tau)_{w\tau}$ of vectors in $M(\tau)$ annihilated by some power of $\mathfrak{m}_{w\tau}$. From [R1, (6.2)] we in fact have

$$\widetilde{\mathfrak{m}}_{w\tau}^n M(\tau)_{w\tau} = 0.$$

Let $\theta \in \widetilde{\mathfrak{m}}_{w\tau}^n$. For $x\tau = w\tau$, we have

$$\theta H_x = \sum_{z\tau=w\tau} H_z \theta_{z,x},$$

for some $\theta_{z,x} \in \mathcal{A}_\tau$, by (1). Since $\theta_{z,x} v_\tau = \theta_{z,x}(\tau)v_\tau$, we have

$$0 = \theta H_x v_\tau = \sum_z H_z \theta_{z,x}(\tau) v_\tau.$$

Since the vectors $H_x v_\tau$ are linearly independent, this shows $\theta_{z,x} \in \widetilde{\mathfrak{m}}_\tau$ for all x, z, so (2) holds.

As for (3), it is clear that

$$F_s \mathcal{H}_{\mathcal{K}}^{w\tau} = \mathcal{H}_{\mathcal{K}}^{sw\tau} \qquad \text{and} \qquad B_s \mathcal{H}_\tau = \mathcal{H}_\tau.$$

If $sw\tau \neq w\tau$, then $\zeta_\alpha \in \mathcal{A}_{w\tau}$, so we have $\zeta_\alpha \mathcal{H}_{w,\tau} \subseteq \mathcal{H}_{w,\tau}$, by (1). Since $F_s = B_s + \zeta_\alpha$, it then follows that

$$F_s \mathcal{H}_{w,\tau} \subseteq \mathcal{H}_{sw,\tau}.$$

The formula in (3) follows from (1.8b), (1.8c).

If $sw\tau = w\tau$, then $F_s \mathcal{H}_{\mathcal{K}}^{w\tau} = \mathcal{H}_{\mathcal{K}}^{w\tau}$ and $\zeta_\alpha \mathcal{H}_{\mathcal{K}}^{w\tau} = \mathcal{H}_{\mathcal{K}}^{w\tau}$, by (1.4a), so $B_s \mathcal{H}_{\mathcal{K}}^{w\tau} \subseteq \mathcal{H}_{\mathcal{K}}^{w\tau}$. Since also $B_s \mathcal{H}_\tau \subseteq \mathcal{H}_\tau$, we have $B_s \mathcal{H}_{w,\tau} \subseteq \mathcal{H}_{w,\tau}$. Therefore

$$B_s H_x = \sum_{z\tau=w\tau} H_z b_{z,x}, \tag{1.9b}$$

for some $b_{z,x} \in \mathcal{A}_\tau$. Now (1.8d) implies that $b_{z,x}$ is also the coefficient of B_z in $B_s H_x$ and that

$$H_x = B_x + \sum_{y\tau\neq w\tau} B_y h_{y,x}, \qquad h_{y,x} \in \mathcal{A}_{w\tau}.$$

By (1.5a) we then have

$$B_s H_x = B_s B_x + \sum_{y\tau\neq w\tau} [\eta_{s,y}(\tau_0) B_{sy} - \zeta_\alpha^y(\tau_0) B_y] h_{y,x}. \tag{1.9c}$$

Comparing coefficients of B_z in (1.9b) and (1.9c), we find that $b_{z,x}$ is in fact the coefficient of B_z in $B_s B_x$, so (4) follows from (1.5a). ∎

2. MATRICES FOR THE PRINCIPAL SERIES

Let $\tau \in \mathbf{T}$, and let $M = M(\tau)$. Recall that the vectors $H_x v_\tau = H_x \otimes 1$, for $x\tau = w\tau$, form a basis of the space $M_{w\tau}$ of vectors in M annihilated by some power of $\mathfrak{m}_{w\tau}$.

Choose a numbering $wW_\tau = \{w_1, \dots, w_n\}$, and form the matrix

$$P_{w\tau} = [p_{w_i, w_j}].$$

Let $D(\theta_1, \dots, \theta_n)$ be the diagonal matrix with diagonal entries $\theta_1, \dots, \theta_n \in \mathcal{K}$. From (1.9)(2), we have

PROPOSITION 2.1. *The matrix of $\theta \in \mathcal{A}_{w\tau}$ acting on $M_{w\tau}$, with respect to the basis $\{H_{w_j} v_\tau : 1 \le i \le n\}$, is*

$$\left[P_{w\tau} D(\theta^{w_1}, \dots, \theta^{w_n}) P_{w\tau}^{-1}\right](\tau).$$

Here each entry of the matrix $P_{w\tau} D(\theta^{w_1}, \dots, \theta^{w_n}) P_{w\tau}^{-1}$ is evaluated at τ.

(2.2) If $s = s_\alpha$ is a simple reflection such that $sw\tau \ne w\tau$, then (1.9)(3) gives the matrix of the map

$$F_s\colon M_{w\tau} \longrightarrow M_{sw\tau},$$

in terms of the bases $\{H_{w_j} v_\tau\}$, $\{H_{sw_j} v_\tau\}$, as

$$\left[P_{sw\tau} D(\eta_{s, w_1}, \dots, \eta_{s, w_1}) P_{w\tau}^{-1}\right](\tau). \tag{2.2a}$$

Since (2.1) gives the matrix of ζ_α on both $M_{w\tau}$ and $M_{sw\tau}$, one can recover from (2.2a) the matrix of T_s acting on $M_{w\tau} \oplus M_{sw\tau}$.

More generally, if $w\tau, s_1 w\tau, \dots, s_m \cdots s_1 w\tau$ are distinct, then at each step we are in the situation of (1.9)(3). Using (1.4c) we get

PROPOSITION 2.3. *Suppose $y = s_m \cdots s_1$ is a reduced expression and that the points $w\tau, s_1 w\tau, \dots, s_m \cdots s_1 w\tau$ are distinct. Then the map $F_y\colon M_{w\tau} \longrightarrow M_{yw\tau}$ is given in terms of the bases $\{H_{w_j} v_\tau\}$, $\{H_{yw_j} v_\tau\}$ by the matrix*

$$\left[P_{yw\tau} D(\eta_{y, w_1}, \dots, \eta_{y, w_n}) P_{w\tau}^{-1}\right](\tau).$$

For $\tau \in \mathbf{T}$, define

$$S_\tau = \{\beta \in \Delta^+ : \ \zeta_\beta \zeta_{-\beta}(\tau) = 0\}. \tag{2.3a}$$

COROLLARY 2.4. *For y, w as in* (2.3), *the map $F_y\colon M_{w\tau} \longrightarrow M_{yw\tau}$ is an isomorphism of vector spaces if and only if $N(y) \cap N(x^{-1}) \cap S_{w\tau} = \varnothing$ for all $x \in wW_\tau$.*

Proof. The stated conditions are equivalent to each η_{y,w_i} being a unit in $\mathscr{A}_\tau$. ∎

Now consider the case essentially opposite to (2.4). That is, assume that $N(y) \cap S_{w\tau} \subseteq N(x^{-1})$ for every $x \in wW_\tau$.

Since

$$\begin{aligned} N(y^{-1}) \cap N(x^{-1}y^{-1}) \cap S_{yw\tau} &= -y[N(y) \cap -y^{-1}N(x^{-1}y^{-1}) \cap S_{w\tau}] \\ &\subseteq -y[N(x^{-1}) \cap -y^{-1}N(x^{-1}y^{-1})] = \varnothing, \end{aligned}$$

we know from Proposition 2.3 and Corollary 2.4 that the map

$$F_{y^{-1}}\colon M_{yw\tau} \longrightarrow M_{w\tau}$$

is an isomorphism, with matrix $[P_{w\tau}D(\eta_{y^{-1},\,yw_1}, \dots, \eta_{y^{-1},\,yw_n})P_{yw\tau}^{-1}](\tau)$. Let

$$F_{y^{-1}}^{-1}\colon M_{w\tau} \longrightarrow M_{yw\tau}$$

be the inverse map.

We have $\eta_{y,\,w_i} \cdot \eta_{y^{-1},\,yw_i} = \mu_y^{w_i}$, where

$$\mu_y = \prod_{\beta \in N(y)} \zeta_\beta \zeta_{-\beta}.$$

Hence, by (2.3), the matrix of $F_y\colon M_{w\tau} \longrightarrow M_{yw\tau}$ is the evaluation at τ of

$$\begin{aligned} &P_{yw\tau}D(\eta_{y,w_1}, \dots, \eta_{y,w_n})P_{w\tau}^{-1} \\ &\quad = \left[P_{yw\tau}D\left(\eta_{y^{-1},\,yw_1}^{-1}, \dots, \eta_{y^{-1},\,yw_n}^{-1}\right)P_{w\tau}^{-1}\right] \cdot \left[P_{w\tau}D\left(\mu_y^{w_1}, \dots, \mu_y^{w_n}\right)P_{w\tau}^{-1}\right]. \end{aligned}$$

The first matrix on the right side is that of $F_{y^{-1}}^{-1}$, and by (2.1) the second is that of $\mu_y \in \mathscr{A}_{w\tau}$ acting on $M_{w\tau}$.

Let $R_{w\tau}(y)$ be the set of all roots $\beta \in \Delta$ such that β and $y\beta$ have opposite signs and $\zeta_\beta(w\tau) = 0$. Then

$$\mu_y = \xi \prod_{\beta \in R_{w\tau}(y)} [e_\beta - e_\beta(w\tau)],$$

where ξ is a unit in $\mathscr{A}_{w\tau}$. We have proved

PROPOSITION 2.5. *Suppose y, w are as in* (2.3) *and that $N(y) \cap S_{w\tau} \subseteq N(x^{-1})$ for every $x \in wW_\tau$. Then there is a unit $\xi \in \mathscr{A}_{w\tau}$ such that the map $F_y\colon M_{w\tau} \longrightarrow M_{yw\tau}$ is given by*

$$F_{y^{-1}}^{-1} \circ \xi \circ \prod_{\beta \in R_{w\tau}(y)} [e_\beta - e_\beta(w\tau)].$$

In particular, the kernel and image of $F_y\colon M_{w\tau} \longrightarrow M_{yw\tau}$ are isomorphic to those of $\prod_{\beta \in R_{w\tau}(y)} [e_\beta - e_\beta(w\tau)]$ acting on $M_{w\tau}$.

3. WEIGHT SPACES IN MORE GENERAL $\mathscr{H}$-MODULES

(3.1) For any finite-dimensional $\mathscr{H}$-module E and $\tau \in \mathbf{T}$, let E_τ be the space of vectors in E which are killed by some power of the maximal ideal $\mathfrak{m}_\tau$. The action of $\mathscr{A}$ on E_τ extends to the localization $\mathscr{A}_\tau$, inducing a left $\mathscr{H}$-module homomorphism

$$\mathscr{H}_\tau \otimes E_\tau \longrightarrow E,$$

denoted $H \otimes v \mapsto Hv$. As recalled in (1.8a), we have a decomposition of right $\mathscr{A}_\tau$-modules,

$$\mathscr{H}_\tau = \bigoplus_{w \in W/W_\tau} \mathscr{H}_{w,\tau},$$

and Proposition 1.9(2) shows that

$$\mathscr{H}_{w,\tau} E_\tau \subseteq E_{w\tau}, \tag{3.1a}$$

with equality if E is generated by E_τ over $\mathscr{H}$.

We note that Nakayama's Lemma extends to $\mathscr{H}_\tau$.

LEMMA 3.2. *Let $H \in \mathscr{H}_\tau$, and suppose $HE_\tau = 0$ for all finite-dimensional $\mathscr{H}$-modules E. Then $H = 0$.*

Proof. For each positive integer ν, define a "higher jet" principal series by

$$M^\nu(\tau) = \mathscr{H}_\tau \otimes_{\mathscr{A}_\tau} (\mathscr{A}_\tau/\widetilde{\mathfrak{m}}_\tau^\nu).$$

Now $\mathscr{H}_\tau$ is right-free over $\mathscr{A}_\tau$, with basis $\{H_x : x \in W\}$, so $M^\nu(\tau)$ is right-free over $\mathscr{A}_\tau/\widetilde{\mathfrak{m}}_\tau^\nu$, with basis $\{H_x \otimes 1 : x \in W\}$. Write $H = \sum_{x\in W} H_x\theta_x$, with $\theta_x \in \mathscr{A}_\tau$. Then in $M^\nu(\tau)$ we have

$$0 = H \otimes 1 = \sum_{x \in W} H_x \otimes \theta_x.$$

It follows that each θ_x belongs to $\widetilde{\mathfrak{m}}_\tau^\nu$ for every ν, hence $\theta_x = 0$ for all x, so $H = 0$. ∎

(3.3) We shall give another basis of $\mathscr{H}_{w,\tau}$ depending on choices of reduced expressions of elements in wW_τ, but having the advantage of being in simple closed form.

For $\tau \in \mathbf{T}$, let Δ_τ^+ denote the set of positive roots β for which ζ_β is not holomorphic at τ. Let $s = s_\alpha$ be a simple reflection. Define

$$F_{s,\tau} = \begin{cases} F_s & \text{if } \alpha \notin \Delta_\tau^+ \\ B_s & \text{if } \alpha \in \Delta_\tau^+. \end{cases}$$

Since $B_s = H_s$ if $\alpha \in \Delta_\tau^+$, we see that

$$F_{s,\tau} = F_{s,s\tau} \in \mathcal{H}_{s,\tau} \cap \mathcal{H}_{s,s\tau}. \tag{3.3a}$$

By (3.1a), we have

$$F_{s,\tau} E_\tau \subseteq E_{s\tau} \tag{3.3b}$$

for any finite-dimensional $\mathcal{H}$-module E.

Let $\mathbf{w} = (s_k, \ldots, s_1)$ be a sequence of simple reflections in W such that $w := s_k \cdots s_1$ has length $l(w) = k$. For $1 \le i \le k$, let $\tau_i = s_i s_{i-1} \cdots s_1 \tau$ and define

$$F_{\mathbf{w},\tau} = F_{s_k,\tau_k} F_{s_{k-1},\tau_{k-1}} \cdots F_{s_1,\tau_1}.$$

If $N(w) \cap \Delta_\tau^+ = \varnothing$, then $F_{\mathbf{w},\tau} = F_w$ by (3.3a) and is therefore independent of the reduced expression for w. However, if $\mathbf{G} = GL_3(\mathbb{C})$, $\tau = (1, t, 1)$ with $t \neq 1$, $\mathbf{w} = (s_1, s_2, s_1)$, then

$$\begin{aligned} F_{\mathbf{w},\tau} &= (B_{s_1} + \zeta_{\alpha_1}) B_{s_2} (B_{s_1} + \zeta_{\alpha_1}) \\ &= F_{s_1 s_2 s_1} - \zeta_{\alpha_1} \zeta_{-\alpha_1} \zeta_{\alpha_1+\alpha_2}. \end{aligned}$$

The first term in the last line is symmetric in 1, 2, but the second is not. Thus, in general, $F_{\mathbf{w},\tau}$ depends on the reduced expression $\mathbf{w}$, not just on w.

LEMMA 3.4. (1) *In $\mathcal{H}_{\mathcal{K}}$ we have*

$$\mathcal{H}_{w\tau} F_{\mathbf{w},\tau} \subseteq \mathcal{H}_\tau.$$

In particular, $F_{\mathbf{w},\tau} \in \mathcal{H}_\tau$.

(2) *If E is a finite-dimensional $\mathcal{H}$-module, we have*

$$F_{\mathbf{w},\tau} E_\tau \subseteq E_{w\tau}.$$

Proof. By induction on $l(w)$, it suffices to prove (1) for $w = s$, a simple reflection. Since both sides are closed under left multiplication by $\mathcal{H}_0$, it suffices to check that $\mathcal{A}_{s\tau} F_{s,\tau} \subseteq \mathcal{H}_\tau$. Let $\theta \in \mathcal{A}_{s\tau}$. We have

$$\theta F_{s,\tau} = \begin{cases} F_s \theta^s & \text{if } \alpha \notin \Delta_\tau^+ \\ B_s \theta^s + \zeta_\alpha(\theta^s - \theta) & \text{if } \alpha \in \Delta_\tau^+. \end{cases}$$

In both cases, the right side belongs to $\mathcal{H}_\tau$. Assertion 2 follows from (3.3b). ∎

PROPOSITION 3.5. (1) *For every $\mathbf{x}$ with $x \in wW_\tau$, we have $F_{\mathbf{x},\tau} \in \mathcal{H}_{w,\tau}$.*

(2) *If we choose one reduced expression $\mathbf{x}$ for each $x \in wW_\tau$, then the collection $\{F_{\mathbf{x},\tau}\}$ is a right $\mathcal{A}_\tau$-basis of $\mathcal{H}_{w,\tau}$.*

Proof. By Lemma 3.4(1), we know that $F_{\mathbf{w},\tau} \in \mathscr{H}_\tau$, so there are $h_y \in \mathscr{H}_{y,\tau}$ such that

$$F_{\mathbf{w},\tau} = h_w + \sum_{y\tau \neq w\tau} h_y.$$

By Proposition 1.9(2), we have $h_y E_\tau \subseteq E_{y\tau}$ for every finite-dimensional $\mathscr{H}$-module E and every y. But then Lemma 3.4(2) implies that $h_y E_\tau = 0$, if $y\tau \neq w\tau$. Then Lemma 3.2 forces these $h_y = 0$. This proves Assertion 1.

Since we insist that $\mathbf{x}$ be a *reduced* expression, Eq. (1.5a) shows that $F_{\mathbf{x},\tau} - B_x$ belongs to the right $\mathscr{A}_\tau$-span of $\{B_y : y < x\}$. Now (2) follows from (1.8d). ∎

COROLLARY 3.6. *If E is a finite-dimensional $\mathscr{H}$-module which is generated by E_τ, then*

$$E_{w\tau} = \sum_{x\tau = w\tau} F_{\mathbf{x},\tau} E_\tau.$$

Thus, if E_τ is a known subspace of a principal series module M, then the matrices in (2.1) and (2.3) can be used to calculate the remaining weight spaces $E_{w\tau}$. In the next two sections we simplify this procedure in a special case.

4. WEIGHT SPACES AND COHOMOLOGY

Let $\tau \in \mathbf{T}$. We assume that the centralizer $\mathbf{G}_\tau$ is connected. Then W_τ is generated by the reflections about the roots in Δ_τ^+, and these roots correspond to a Borel subgroup $\mathbf{B}_\tau \subset \mathbf{G}_\tau$. In this section we review some well-known facts about the cohomology of the flag variety $\mathscr{B}_\tau = \mathbf{G}_\tau / \mathbf{B}_\tau$ (cf. [BGG]).

(4.1) Recall that $\mathfrak{m}_\tau$ is the maximal ideal in $\mathscr{A}$ at τ. The action of W_τ on $\mathscr{A}$ preserves $\mathfrak{m}_\tau$, and we let I_τ be the ideal in $\mathscr{A}$ generated by the W_τ-invariants in $\mathfrak{m}_\tau$. The quotient $\mathscr{A}/I_\tau$ is naturally isomorphic to the cohomology ring $H^*(\mathscr{B}_\tau)$. More precisely, $H^*(\mathscr{B}_\tau)$ is commutative, and we have a natural ring isomorphism

$$j_\tau : \mathscr{A}/I_\tau \xrightarrow{\simeq} H^*(\mathscr{B}_\tau) \tag{4.1a}$$

such that

$$j_\tau(e_\lambda) = e_\lambda(\tau)\exp(c_\lambda) = e_\lambda(\tau)\left[1 + c_\lambda + \frac{1}{2!}c_\lambda^2 + \cdots\right],$$

where c_λ is the first Chern class of the line bundle L_λ on $\mathscr{B}_\tau$ induced by e_λ. The group W_τ acts on both sides of (4.1a), and j_τ is W_τ-equivariant. Both sides of (4.1a) are isomorphic to the regular representation of W_τ.

It follows from (4.1a) that the homology $H_*(\mathcal{B}_\tau)$ is a module over $\mathcal{A}/I_\tau$, via the cap-product. This can be made more explicit: Let Σ_τ be the base of Δ_τ^+ and let $\mathcal{S}$ be the ring of polynomials in variables h_β, for $\beta \in \Sigma_\tau$. For an arbitrary positive root $\beta \in \Delta_\tau^+$, we define h_β as follows: If $\check{\beta} = \sum_{\alpha\in\Sigma_\tau} c_\alpha \check{\alpha}$, then $h_\beta = \sum c_\alpha h_\alpha$.

For $\lambda \in X^*(\mathbf{T})$, let ∂_λ be the derivation of $\mathcal{S}$ determined by the condition

$$\partial_\lambda(h_\beta) = \langle \lambda, \check{\beta}\rangle.$$

Then $\mathcal{S}$ is a locally finite $\mathcal{A}$-module on which e_λ acts by the operator

$$e_\lambda(\tau)\exp(\partial_\lambda) = e_\lambda(\tau)\left[1 + \partial_\lambda + \frac{1}{2!}\partial_\lambda^2 + \cdots\right].$$

Let $\mathbf{H} \subset \mathcal{S}$ be the space of polynomials annihilated by the W_τ-invariants in $\mathfrak{m}_\tau$. Dual to (4.1a), we have a W_τ-equivariant isomorphism

$$H_*(\mathcal{B}_\tau) \longrightarrow \mathbf{H},$$

sending the fundamental class $[\mathcal{B}_\tau]$ to the polynomial

$$\Pi_\tau = \prod_{\beta\in\Delta_\tau^+} h_\beta,$$

such that the cap product by c_λ on $H_*(\mathcal{B}_\tau)$ corresponds to the operator ∂_λ on $\mathbf{H}$.

In view of (2.5), we are interested in the kernels and images of the following kinds of elements of $\mathfrak{m}_\tau$ acting on $\mathbf{H}$. Let $R \subseteq \Delta$ be a set of roots. Put

$$m_R = \prod_{\beta\in R}[e_\beta - e_\beta(\tau)], \qquad \partial_R = \prod_{\beta\in R}\partial_\beta.$$

As operators on $\mathbf{H}$ we have

$$m_R = c\partial_R + \cdots,$$

where c is a nonzero constant, and $\cdots$ indicates operators of order larger than $\#R$. It follows that the kernel and image of m_R, after grading according to the filtration of $\mathbf{H}$ by increasing degree, become isomorphic to those of ∂_R. In particular, the nullity and rank of m_R are equal to those of ∂_R.

(4.2) Now suppose that E is an $\mathcal{H}$-module which can be realized as a subquotient of some principal series module $M(\tau')$.

PROPOSITION 4.3. *In this situation, the ideal I_τ annihilates E_τ. Hence E_τ is a module over $H^*(\mathcal{B}_\tau)$, via the isomorphism* (4.1a).

Proof. We may assume that $E_\tau \neq 0$. Since E is a subquotient of $M(\tau')$, we have $w\tau' = \tau$ for some $w \in W$, and moreover it suffices to prove the result for $E = M(\tau')$. Suppose $\theta \in \mathfrak{m}_\tau$ is fixed by W_τ. Then $\theta^w \in \mathfrak{m}_{\tau'}$ is fixed by $W_{\tau'}$. Let

$$[\theta_{x,z}]_{x,z \in wW_{\tau'}}$$

be the matrix of θ acting on $M(\tau')_\tau$, as in (2.1). So $\theta_{x,z}$ is the evaluation at τ' of

$$\sum_{y \in W_{\tau'}} p_{x,wy} \theta^{wy} r_{wy,z} = \theta^w \sum_{y \in W_{\tau'}} p_{x,wy} r_{wy,z} = \theta^w \delta_{x,z},$$

where $[\delta_{x,z}]$ is the identity matrix. Hence $\theta_{x,z} = \theta^w(\tau')\delta_{x,z} = 0$. ∎

PROPOSITION 4.4. *Let $M = M(\tau)$ be a principal series module and let $w \in W$. Then the $\mathscr{A}$-module $M_{w\tau}$ is isomorphic to $H_*(\mathscr{B}_{w\tau})$ if and only if it is cyclic.*

Proof. By Section 4.1 and Poincaré duality, the $\mathscr{A}$-module $H_*(\mathscr{B}_{w\tau})$ is cyclic and is generated by the fundamental class of $\mathscr{B}_{w\tau}$. Conversely, suppose we have a surjective $\mathscr{A}$-homomorphism $\mathscr{A} \longrightarrow M_{w\tau}$. By (4.3) we then have a surjection

$$\mathscr{A}/I_{w\tau} \longrightarrow M_{w\tau}.$$

By Section 4.1, the dimension of $\mathscr{A}/I_{w\tau}$ is $|W_\tau|$, and the same is true of $M_{w\tau}$ (cf. [R1, (2.2)]). Finally, $I_{w\tau}$ is the annihilator of $[\mathscr{B}_{w\tau}]$, so $\mathscr{A}/I_{w\tau} \simeq H_*(\mathscr{B}_{w\tau})$ as $\mathscr{A}$-modules. ∎

5. STANDARD SINGULARITIES

Each $\tau \in \mathbf{T}$ has a canonical polar decomposition $\tau = \tau_e \tau_h$, such that for all $\lambda \in X^*(\mathbf{T})$ we have $|e_\lambda(\tau_e)| = 1$, $e_\lambda(\tau_h) > 0$. For any subset $J \subseteq \Sigma$, let W_J be the corresponding standard parabolic subgroup of W, generated by reflections from J. The centralizer W_{τ_h} of τ_h in W is conjugate to W_J for some J.

DEFINITION 5.1. We say that τ has *standard singularity of type J*, for $J \subseteq \Sigma$, if the centralizer $\mathbf{G}_\tau$ is connected and $W_{\tau_h} = W_J$.

If $\mathbf{G}$ has a simply-connected derived group, then $\mathbf{G}_\tau$ is always connected (Steinberg's Theorem). In this case, every W-orbit in $\mathbf{T}$ contains an element with standard singularity. For example, we can choose τ in its W-orbit so that $e_\beta(\tau_h) \leq 1$ for all $\beta > 0$. Then τ has standard singularity.

If τ has standard singularity, we will show that all weight spaces in $M(\tau)$ are W-twists of one another, in the following sense: If we have an $\mathscr{A}$-module

$$\pi\colon \mathscr{A} \longrightarrow \mathrm{End}(N)$$

and $w \in W$, then $\theta \in \mathscr{A}$ acts on the twisted module wN via $\pi(\theta^w)$.

PROPOSITION 5.2. *Suppose that τ has standard singularity, and $M = M(\tau)$. If $\tau' \in W\tau$, then there exists $w \in W$ such that $\tau' = w\tau$ and $M_{w\tau} \simeq wM_\tau$, as $\mathscr{A}$-modules.*

Remarks. Note that $M_{w\tau}$ depends only on $w\tau$, and we will see later that the isomorphism class of the $\mathscr{A}$-module wM_τ is also independent of the choice of w in its W_τ-coset. At this stage, however, we choose a suitable w, and the isomorphism in (5.2) will then be given by F_w.

The result is false without the hypothesis of standard singularity, as can be seen from the example in $\mathbf{G} = GL_3(\mathbb{C})$, where $\tau = (1, q, 1)$. Then M_τ splits into two one-dimensional $\mathscr{A}$-modules, whereas $M_{(q,1,1)}$ and $M_{(1,1,q)}$ are indecomposable (cf. [R1, (4.6), (15.5)]).

Proof. By [R1, (10.13)], we can choose a sequence of simple reflections $s_1, s_2, \ldots, s_k$ such that the points

$$\tau,\ s_1\tau, \ldots, s_k \cdots s_1\tau = \tau'$$

are distinct and the expression $s_k \cdots s_1$ is reduced. It follows that

$$F_{s_k \cdots s_1} = F_{s_k} \cdots F_{s_1} \in \mathscr{H}_\tau.$$

By induction on k, we show that $F_{s_k \ldots s_1}$ gives the desired isomorphism. Let $w_1 = s_{k-1} \cdots s_1$, and let $s = s_k = s_\alpha$. Assume that

$$F_{w_1}\colon M_\tau \longrightarrow M_{w_1\tau}$$

is bijective. We want to show that

$$F_s\colon M_{w_1\tau} \longrightarrow M_{sw_1\tau}$$

is also bijective. By (2.4), it suffices to assume that $\zeta_\alpha\zeta_{-\alpha}(w_1\tau) = 0$ and then to show that $sx > x$ for every $x \in w_1W_\tau$.

Note that $W_\tau \subseteq W_J$, since the polar decomposition of τ is canonical. Let $W^J = \{y \in W : yJ \subset \Delta^+\}$. Write $x = yz$, where $y \in W^J$ and $z \in W_J$. Since $\zeta_\alpha\zeta_{-\alpha}(w_1\tau) = 0$, we have

$$1 \neq |e_\alpha(w_1\tau)| = |e_\alpha(x\tau)| = e_\alpha(x\tau_h) = e_\alpha(y\tau_h).$$

It follows that $y^{-1}\alpha$ does not belong to the span of J, so $sy \in W^J$. Now from $sw_1 > w_1$ it follows (cf. [J, 2.22b]) that $sy > y$. Since

$$N(x^{-1}) = N(y^{-1}) \cup yN(z^{-1})$$

and $N(z^{-1})$ is contained in the span of J, we cannot have $\alpha \in N(x^{-1})$, so $sx > x$ as desired. ∎

PROPOSITION 5.3. *Assume that τ has standard singularity. Let $M = M(\tau)$. Then every weight space $M_{w\tau}$ is cyclic over $\mathscr{A}$, hence, by* (4.4), *it is isomorphic to $H_*(\mathscr{B}_{w\tau})$.*

Proof. Suppose that τ has standard singularity of type J. By (5.2), it suffices to prove that M_τ is cyclic. Let

$$M_J = \bigoplus_{z \in W_J/W_\tau} M_{z\tau}. \tag{5.3a}$$

Let $\mathscr{H}_{J,0} \subset \mathscr{H}_0$ be the subalgebra generated by T_{s_α} for $\alpha \in J$, and let $\mathscr{H}_J$ denote the subalgebra

$$\mathscr{H}_J := \mathscr{H}_{J,0} \tilde{\otimes} \mathscr{A} \subseteq \mathscr{H}.$$

Then M_J is a principal series module over $\mathscr{H}_J$, generated by v_τ, and is irreducible by Kato's criterion [K, Theorem 2.2], since $\zeta_\alpha(\tau) \neq 0$ for all $\alpha \in \Delta_J$, as noted in the proof of Proposition 5.2. (Note that condition (ii) in Kato's theorem holds automatically by our assumption that $\mathbf{G}_\tau$ is connected.) Therefore the vector $B_J = B_{w_J} v_\tau$ generates M_J over $\mathscr{H}_J$.

For $\alpha \in J$ we have

$$B_{s_\alpha} B_J = -(1 + q_\alpha) B_J,$$

by (1.5a). Thus, B_J is the unique vector in M_J up to scalar, which affords the sign character of $\mathscr{H}_{J,0}$, so $\mathscr{A} B_J = M_J$. Let $B_{J,\tau}$ be the projection of B_J to M_τ according to decomposition (5.3a). Then $M_\tau = \mathscr{A} B_{J,\tau}$. ∎

COROLLARY 5.4. *If τ has standard singularity and $w \in W$, the isomorphism class of the $\mathscr{A}$-module wM_τ depends only on $w\tau$.*

Proof. Let $\mathscr{A} \rtimes W_\tau$ be the tensor product of $\mathscr{A}$ and the group algebra of W_τ, with multiplication rule

$$\theta \cdot x = x \cdot \theta^x.$$

By the naturality of Chern classes, the actions of $\mathscr{A}$ and W_τ on $H_*(\mathscr{B}_\tau)$ combine to make the latter an $\mathscr{A} \rtimes W_\tau$ module. It follows that the isomorphism class of the $\mathscr{A}$-module $H_*(\mathscr{B}_\tau)$ is invariant under twisting by any $x \in W_\tau$. The same holds for M_τ, by Proposition 5.3. ∎

(5.5) We are ready to complete the proof of the theorem stated in the Introduction. Suppose that $\tau \in \mathbf{T}$ has standard singularity of type J.

Let w^J be the longest element of W^J, and set $\bar{\tau} := w^J \tau$. Then $\bar{\tau}$ has standard singularity of type $\bar{J} := w^J J$. Let $M = M(\bar{\tau})$. As in the proof of (5.3), the sum of weight spaces

$$\bigoplus_{z \in W_J/W_\tau} M_{z\tau}$$

is an irreducible principal series module over $\mathscr{H}_J$. It follows that there is a unique-up-to-isomorphism simple $\mathscr{H}$-module $E = E(\tau)$ such that $E_\tau \neq 0$, namely, E is the unique simple quotient of $M(\tau)$. By [R1, (3.8)], E is also the unique submodule of M. In fact, E is the submodule of M generated by M_τ and $E_{z\tau} = M_{z\tau}$ for all $z \in W_J$.

EXAMPLES. If $e_\beta(\tau_h) \geq 1$ for all $\beta > 0$, then $E(\tau)$ has standard singularity. And by [K, Theorem 2.4], $E(\tau)$ is the unique constituent of $M(\tau)$ containing the trivial character of $\mathscr{H}_0$. (This requires $q_0 \leq q_\beta$ in (1.2a).) Likewise, if $e_\beta(\tau_h) \leq 1$ for all $\beta > 0$, then $E(\tau)$ has standard singularity and is the unique constituent of $M(\tau)$ containing the sign character of $\mathscr{H}_0$. If $\mathbf{G} = GL_4(\mathbb{C})$ and $\tau = (q, q, 1, q^2)$, then τ has standard singularity and $E(\tau)$ is the full induced module from trivial$\otimes$sign on the $A_1 \times A_1$ parabolic subalgebra, hence $E(\tau)$ contains neither the trivial nor the sign characters of $\mathscr{H}_0$.

We want to compute the weight space $E_{w\tau}$, for $w \in W$. We may and shall choose w to have minimal length in its W_τ-coset. Write $w = yz$ with $y \in W^J$, $z \in W_J$.

By (3.6), we have

$$E_{w\tau} = \sum_{x\tau = w\tau} F_{\mathbf{x},\tau} E_\tau$$

for fixed choices of reduced expressions $\mathbf{x}$. Now $x\tau = w\tau$ means $x = yzu$ for some $u \in W_\tau \subset W_J$, and associating zu to x gives a bijection

$$\{x \in W : x\tau = w\tau\} \leftrightarrow \{v \in W_J : v\tau = z\tau\}.$$

We may choose reduced expressions $\mathbf{y}$, $\mathbf{v}$, such that

$$\mathbf{x} = (\mathbf{y}, \mathbf{v})$$

is also reduced. Moreover, $F_{\mathbf{y},\tau} = F_y$, since $N(y) \cap \Delta_\tau^+ \subseteq N(y) \cap \Delta_J^+ = \varnothing$. Thus

$$E_{w\tau} = \sum_{x\tau = w\tau} F_{\mathbf{x},\tau} E_\tau = F_y \sum_{v\tau = z\tau} F_{\mathbf{v},\tau} E_\tau = F_y E_{z\tau} = F_y M_{z\tau}. \tag{5.5a}$$

We want to verify the hypotheses of (2.5). The data $(\tau, w, w\tau, y)$ of (2.5) are here $(\bar{\tau}, z(w^J)^{-1}, z\tau, y)$.

Let $y = s_k \cdots s_1$ be a reduced expression. By the minimality of $l(w)$, the points $z\tau, s_1 z\tau, \ldots, s_k \cdots s_1 z\tau$ are distinct. It will suffice to show that

$$N(y) \subseteq N(x^{-1}) \tag{5.5b}$$

for all

$$x \in z(w^J)^{-1} W_{\bar{\tau}}.$$

This last is $zW_\tau(w^J)^{-1} \subseteq W_J(w^J)^{-1}$, so $x^{-1} \in w^J W_J = W_{\bar{J}} w^J$. Write $x^{-1} = tw^J$ with $t \in W_{\bar{J}}$.

Now

$$N(y) = \{s_1 \cdots s_{i-1}\alpha_i : 1 \le i \le k\}.$$

If $y \neq w^J$, one checks, by downward induction on the length of y, that there are simple reflections $s_{k+1}, \ldots, s_m$ such that

$$y < s_{k+1}y < s_{k+2}s_{k+1}y < \cdots < s_m \cdots s_{k+1}y = w^J.$$

Then $w^J = s_m s_{m-1} \cdots s_1$ is reduced and

$$\{s_m s_{m-1} \cdots s_i \alpha_i : 1 \le i \le m\} = -N((w^J)^{-1}) = -[\Delta^+ \backslash \Delta^+_{\bar{J}}].$$

For $1 \le i \le k$ we have

$$x^{-1}s_1 \cdots s_{i-1}\alpha_i = ts_m s_{m-1} \cdots s_i \alpha_i \in -t[\Delta^+ \backslash \Delta^+_{\bar{J}}] \subseteq -\Delta^+,$$

since $t \in W_{\bar{J}}$. Hence (5.5b) holds, and we can apply Proposition 2.5. Recalling (5.5a), this tells us that the $\mathscr{A}$-module $E_{w\tau}$ is the twist by y of the image of

$$\prod_{\beta \in R_{z\tau}(y)} e_\beta - e_\beta(z\tau) \in \mathfrak{m}_{z\tau} \tag{5.5c}$$

acting on $M_{z\tau}$, where $R_{z\tau}(y)$ is the set of roots $\beta \in \Delta$ for which β and $y\beta$ have opposite signs and $\zeta_\beta(z\tau) = 0$.

Recalling Proposition 5.3 and the remarks at the end of Section 4.1, we have proved our main result:

THEOREM 5.6. *Suppose τ has standard singularity of type J and let E be the unique simple $\mathscr{H}$-module such that $E_\tau \neq 0$. Then the weight space $E_{w\tau}$ may be computed as follows. Choose w to have minimal length in its W_τ-coset. Write $w = yz$ with $y \in W^J$, $z \in W_J$. Then $E_{w\tau}$ is isomorphic to the twist by y of the $\mathscr{A}$-submodule of $H_*(\mathscr{B}_{z\tau})$ generated by*

$$j_{z\tau}\left(\prod_{\beta \in R_{z\tau}(y)} e_\beta - e_\beta(z\tau)\right) \cap [\mathscr{B}_{z\tau}].$$

Moreover, the dimension of $E_{w\tau}$ equals the dimension of the $\mathscr{A}$-submodule of $\mathscr{S}$ generated by

$$\left(\prod_{\beta \in R_{z\tau}(y)} \partial_\beta\right)\Pi_{z\tau}.$$

Remark 5.7. When computing $\left(\prod_{\beta\in R_{z\tau}(y)}\partial_\beta\right)\Pi_{z\tau}$, one can replace $R_{z\tau}(y)$ by a set of positive roots, namely the set

$$N(y)\cap S_{z\tau}=\{\beta\in N(y):\ \zeta_\beta(z\tau)\zeta_{-\beta}(z\tau)=0\}.$$

For, if $\beta\in R_{z\tau}(y)$, then exactly one of $\pm\beta$ belongs to $N(y)\cap S_{z\tau}$ and vice versa, so

$$\prod_{\beta\in R_{z\tau}(y)}\partial_\beta=\pm\prod_{\beta\in N(y)\cap S_{z\tau}}\partial_\beta.$$

6. REMARKS ON THE MATRIX P_τ

We give here two formulas involving the matrix

$$P_\tau=[p_{u,\,v}]_{u,\,v\in W_\tau},$$

reminiscent of identities between Kazhdan–Lusztig polynomials.

PROPOSITION 6.1. *Assume $W_\tau\subseteq W_J$, for some $J\subseteq\Sigma$. Let $w\in W$, and write $w=yz$ with $y\in W^J$, $z\in W_J$. Then $P_{w\tau}=P_{z\tau}$.*

Proof. Replacing τ by $z\tau$, we may assume $z=1$ and $w\in W^J$. Suppose $sw<w$. Then $sw\in W^J$ as well. It suffices to show that

$$p_{swu,\,swv}=p_{wu,\,wv}$$

for all $u,v\in W_\tau$. Since $w\in W^J$, $u\in W_J$, we have

$$N(w^{-1})\subseteq N((wu)^{-1}),$$

so $swu<wu$ for all $u\in W_\tau$. Using the recursion (1.6) we have

$$p_{wu,wv}=[\zeta_\alpha^{swv}-\zeta_\alpha^{wu}(\tau_0)]p_{wu,\,swv}+p_{swu,\,swv}.$$

It suffices to show that $wu\not\leq swv$. But if $wu\leq swv$, then $w\leq wu\leq swv$, and since $w\in W^J$, no reduced expression for w can end in a root from J. Hence $w<sw$, a contradiction. ∎

The second formula is an inversion formula for P_τ, assuming the stronger condition $W_\tau=W_J$. Then we may as well assume that $W_\tau=W$ and consider the matrix

$$P=[p_{z,\,x}]_{z,\,x\in W}.$$

Denote the inverse matrix coefficients by

$$P^{-1}=[p^{z,\,x}]_{z,\,x\in W}.$$

PROPOSITION 6.2. *We have*

$$p^{z,x} = \epsilon(xz)p_{w_0x,\,w_0z},$$

where ϵ is the sign character of W and w_0 is the longest element of W.

Proof. Recalling the definition of $p_{z,x}$, we have to prove the following identity in $\mathscr{H}_{\mathscr{K}}$, for every $x \in W$:

$$B_x = \sum_{z\in W} F_z \epsilon(xz)p_{w_0x,w_0z}. \tag{6.2a}$$

By induction on length, we may assume that (6.2a) holds for x and let $s = s_\alpha$ be a simple reflection such that $sx > x$. Then, by (1.5),

$$\begin{aligned}
B_{sx} &= \zeta_\alpha^x(\tau_0)B_x + B_sB_x \\
&= \zeta_\alpha^x(\tau_0)\sum_{z\in W} F_z\epsilon(xz)p_{w_0x,\,w_0z} + \sum_{z\in W}(F_s - \zeta_\alpha)F_z\epsilon(xz)p_{w_0x,w_0z} \\
&= \zeta_\alpha^x(\tau_0)\sum_{z\in W} F_z\epsilon(xz)p_{w_0x,\,w_0z} + \sum_{z\in W}[F_{sz}\eta_{\alpha,\,z} - F_z\zeta_\alpha^z]\epsilon(xz)p_{w_0x,\,w_0z} \\
&= \sum_{z\in W} F_z\epsilon(sxz)\big\{[\zeta_\alpha^z - \zeta_\alpha^x(\tau_0)]p_{w_0x,\,w_0z} + \eta_{\alpha,\,sz}p_{w_0x,\,w_0sz}\big\},
\end{aligned}$$

so we must show that the expression in $\{\ ,\ \}$ is $p_{w_0sx,\,w_0z}$. For $sz < z$, this is obtained by applying the recursion (1.6) to $p_{w_0x,\,w_0sz}$, then using the identity $\zeta_\beta + \zeta_{-\beta} = 1 + q_\beta$. Suppose $sz > z$. Applying (1.6) to $p_{w_0x,\,w_0z}$ and p_{w_0sx,w_0z} gives

$$\begin{aligned}
p_{w_0x,\,w_0z} &= [\zeta_\alpha^z - \zeta_{-\alpha}^x(\tau_0)]p_{w_0x,\,w_0sz} + p_{w_0sx,\,w_0sz}, \\
[\zeta_\alpha^z - \zeta_\alpha^x(\tau_0)]p_{w_0sx,\,w_0sz} &= p_{w_0sx,\,w_0z} - [\zeta_\alpha\zeta_{-\alpha}]^x(\tau_0)p_{w_0x,\,w_0sz}.
\end{aligned}$$

The conclusion follows as in the previous case. ∎

Continue to assume that $W_\tau = W_J$. Let w_J be the longest element of W_J. Let

$$\Theta = [\theta_x]_{x\in W_J}$$

be a diagonal matrix indexed by W_J, with diagonal entries $\theta_x \in \mathscr{A}_\tau$, and let

$$\pi = [\pi_{x,\,z}]_{x,\,z\in W_J} = P_\tau \Theta P_\tau^{-1}.$$

We have seen that generators of $\mathscr{H}$ act on the principal series $M(\tau)$ by matrices of the form $\pi(\tau)$.

COROLLARY 6.3. (1)

$$\pi_{z,\,x} = \sum_{x\le y\le z} \epsilon(yz)\theta_y p_{x,\,y}p_{w_Jz,\,w_Jy}.$$

(2)

$$\pi_{w_J z, w_J x} = \epsilon(xz)(\pi^{w_J})_{x, z},$$

where π^{w_J} is obtained from π by replacing Θ by Θ^{w_J}.

(3)

$$\pi_{e, w_J} = \epsilon(w_J)\zeta_{w_J} \sum_{y \in W_J} \epsilon(y)\theta_y.$$

If $\theta_y = e_\lambda^y$, where $\lambda \in X^*(\mathbf{T})$ is J-dominant, the sum in (3) is $\nu_{w_J}\chi_J(\lambda)$, where ν_{w_J} is the numerator of ζ_{w_J}, and $\chi_J(\lambda)$ is the character of the representation of the Levi subgroup L_J, with highest weight λ. Since τ belongs to the center of L_J, it follows that the matrix coefficient $\pi_{e, w_J}(\tau)$ is given by the Weyl dimension formula.

7. AN EXAMPLE

Let $\mathbf{G} = F_4(\mathbb{C})$, with simple roots labelled 1—2 $\Leftarrow$ 3—4. For $c > 0$, consider the affine Hecke algebra $\mathscr{H}^c$ attached to $\mathbf{G}$, with parameters $q_0 = q_1 = q_2 = q > 1$, $q_3 = q_4 = q^c$.

Let k be a nonarchimedean local field of residue cardinality q. Let $\mathscr{I}$ be an Iwahori subgroup of the p-adic Chevalley group $F_4(k)$. Then $\mathscr{H}^1$ is the $\mathscr{I}$-spherical Hecke algebra of $F_4(k)$. The irreducible admissible representations of $F_4(k)$ containing a fixed vector under $\mathscr{I}$ correspond bijectively to the finite-dimensional irreducible representations $\mathscr{H}^1$, via the functor $V \mapsto V^{\mathscr{I}}$; see [B].

Although it is useful to let the parameter c vary continuously, we are most interested in $\mathscr{H}^4$, which arises as follows. Let P be the parahoric subgroup in $E_8(k)$ of type D_4, and let σ be the unique cuspidal unipotent representation of the reductive quotient of P. We view σ as a representation of P. Then $\mathscr{H}^4$ is isomorphic to the algebra of smooth compactly supported functions $f\colon E_8(k) \longrightarrow \operatorname{End}(\sigma)$, such that $f(pgp') = \sigma(p)f(g)\sigma(p')$ for all $g \in G$, $p, p' \in P$. The irreducible admissible representations V of $E_8(k)$ containing σ upon restriction to P correspond bijectively to the finite-dimensional irreducible representations of $\mathscr{H}^4$, via the functor $V \mapsto V^\sigma := \operatorname{Hom}_P(\sigma, V)$; see [L1, M].

If $V^{\mathscr{I}} \neq 0$, or $V^\sigma \neq 0$, respectively, then V is square integrable if and only if all weights τ in $V^{\mathscr{I}}, V^\sigma$ have the property $|e_\lambda(\tau)| < 1$ for every dominant weight λ of $\mathbf{G} = F_4(\mathbb{C})$. From [KL] we know that $F_4(k)$ has exactly 18 square integrable representations of $F_4(k)$ with $V^{\mathscr{I}} \neq 0$. Likewise, in [R4], there are listed 18 square integrable representations V of $E_8(k)$ with $V^\sigma \neq 0$. (This list is now known to be complete.) Most of the corresponding representations V^σ of $\mathscr{H}^4$ have standard singularity, in the sense of Definition 5.1. One of these, labelled $[A_1E_7, -3]$ in [R4], cannot be analyzed by the results in [R1]. To describe its weights, we write $\tau = [t_1, t_2, t_3, t_4] \in \mathbf{T}$, a maximal torus of $F_4(\mathbb{C})$, where $e_{\alpha_i}(\tau) = q^{-t_i}$ (all t_i will be real, in this example).

Suppose $c > 2$. Consider the weights

$$\tau = [0, 1, 0, c-2], \qquad \tau' = [1-c, 0, c, 0] \in \mathbf{T}.$$

These have standard singularities of type $J = \{\alpha_1, \alpha_3\}$, $J' = \{\alpha_2, \alpha_4\}$, respectively.

Each W-orbit in $\mathbf{T}$ forms a graph, with an edge between τ_1 and τ_2 iff there is a simple root α such that $s_\alpha \tau_1 = \tau_2$ and $\zeta_\alpha(\tau_1)\zeta_{-\alpha}(\tau_1)$ is finite nonzero. The weight multiplicities are constant on the components of the graph [R1, 3.6]. In this case, τ and τ' belong to the same component,

$$\tau = [0, 1, 0, c-2] \overset{4}{—} [0, 1, c-2, 2-c] \overset{3}{—} [0, c-1, 2-c, 0]$$
$$\overset{2}{—} [c-1, 1-c, c, 0] \overset{1}{—} [1-c, 0, c, 0] = \tau' .$$

Hence there is a unique simple $\mathscr{H}^c$-module E^c containing τ, τ', and these weights have multiplicity $4 = |W_\tau|$ in E^c. Now, E^4 is our module $[A_1E_7, -3]$. We will use Theorem (5.6) to find the weights in E^c. This will show, among other things, that E^4 is square integrable. Since $\tau = \tau_h$, we have $z = 1$ in Theorem 5.6.

We have

$$\Pi_\tau = h_{\alpha_1} h_{\alpha_3}, \qquad \Pi_{\tau'} = h_{\alpha_2} h_{\alpha_4}.$$

Since

$$\partial_{\alpha_2}\Pi_\tau = -h_{\alpha_1} - h_{\alpha_3} \neq 0, \qquad \partial_{\alpha_3}\Pi_{\tau'} = -2h_{\alpha_4} - h_{\alpha_2} \neq 0,$$

we find that E^c contains the components of $s_2\tau$ and $s_3\tau'$, namely,

$$[1, 1, -2, c] \overset{3}{—} [1, -1, 2, c-2] \overset{4}{—} [1, -1, c, 2-c]$$

and

$$[c-1, 1, -c, c] \overset{1}{—} [1-c, c, -c, c] \overset{2}{—} [1, -c, c, c],$$

and these weights have multiplicity two in E^c.

Since $N(s_1s_2) \cap S_\tau = \{\alpha_2, \alpha_1 + \alpha_2\}$ and $N(s_4s_3) \cap S_\tau = \{\alpha_3, \alpha_3 + \alpha_4\}$, we have

$$\partial_{\alpha_1+\alpha_2}\partial_{\alpha_2}\Pi_\tau = -1 + 1 = 0, \qquad \partial_{\alpha_3+\alpha_4}\partial_{\alpha_3}\Pi_{\tau'} = -2 + 2 = 0,$$

so the weights $s_1 s_2 \tau$ and $s_4 s_3 \tau'$ do not appear in E^c. On the other hand,

$$N(s_4 s_3 s_2) \cap S_\tau = \{\alpha_2, 2\alpha_2 + \alpha_3 + \alpha_4\}$$

and

$$\partial_{2\alpha_2+\alpha_3+\alpha_4} \partial_{\alpha_2} \Pi_\tau = 3 \neq 0,$$

so $s_4 s_3 s_2 \tau$ is a weight in E^c with multiplicity one. Its component is

$$\begin{tikzcd}[column sep=small]
s_1 s_2 s_3 \tau' = [-1, 1-c, c, c] \arrow[r, dash, "2"] & {[-c, c-1, 2-c, c]} \arrow[r, dash, "3"] \arrow[d, dash, "1"] & {[-c, 1, c-2, 2]} \arrow[r, dash, "4"] \arrow[d, dash, "1"] & {[-c, 1, c, -2]} \arrow[d, dash, "1"] & \\
& {[c, -1, 2-c, c]} \arrow[r, dash, "3"] & {[c, 1-c, c-2, 2]} \arrow[r, dash, "4"] \arrow[d, dash, "2"] & {[c, 1-c, c, -2]} \arrow[d, dash, "2"] & \\
& & {[1, c-1, -c, 2]} \arrow[r, dash, "4"] & {[1, c-1, 2-c, -2]} \arrow[d, dash, "3"] & \\
& & & {[1, 1, c-2, -c]} & = s_4 s_3 s_2 \tau.
\end{tikzcd}$$

Continuing in this way, we find no more weights in E^c. Writing each of the fundamental dominant weights as linear combinations of simple roots, one verifies the square-integrability condition for $c = 4$.

The weights provide additional information about the corresponding representation V of $E_8(k)$ that is needed in [R4]. The co-roots $\check{\alpha}_i$ are naturally associated to simple roots of E_8 outside the Levi subgroup L of type D_4, by means of the diagram

$$\begin{tikzcd}
\check{\alpha}_1 \arrow[r, dash] & \check{\alpha}_2 \arrow[r, dash] & \check{\alpha}_3 \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] & \check{\alpha}_4 \\
& & & & \bullet & &
\end{tikzcd}$$

Let L^{ad} be the adjoint group of L. Since L has a connected center, it follows from [B2, 15.7] that the natural homomorphism $L \longrightarrow L^{\mathrm{ad}}$ is surjective on k-rational points. Now σ may be viewed as a representation of a hyperspecial maximal compact subgroup of $L^{\mathrm{ad}}(k)$. Via compact induction, we get an irreducible supercuspidal representation $[\sigma]$ of $L^{\mathrm{ad}}(k)$. We view $[\sigma]$ as a representation of $L(k)$ via the surjection $L(k) \longrightarrow L^{\mathrm{ad}}(k)$.

The torus $\mathbf{T}$ may be identified with the set of unramified characters of $L(k)$. For $\tau \in \mathbf{T}$, with $\bar{\tau}$ as in Section 5.5, we have [R3, (6.1)]

$$(\mathrm{Ind}_{Q(k)}^{E_8(k)} [\sigma] \otimes \tau)^\sigma \simeq M(\bar{\tau}),$$

where $M(\bar{\tau})$ is the principal series module for $\mathscr{H}^4$, as in Section 1, and Q is the standard parabolic subgroup of E_8 with Levi L. Since E^4 is the unique irreducible submodule of $M(\bar{\tau})$ (see Section 5.5), we know that V is the unique irreducible subrepresentation of $\mathrm{Ind}_{Q(k)}^{G(k)} [\sigma] \otimes \tau$. But the weights say more: The map $F_{s_1}\colon E^4_{s_2s_3\tau'} \longrightarrow E^4_{s_1s_2s_3\tau'}$ has a one-dimensional kernel U. Since F_{s_3} and F_{s_4} kill $E^4_{s_2s_3\tau'}$, this kernel is invariant under the parabolic subalgebra $\mathscr{H}^4_I \subset \mathscr{H}^4$, where $I = \{\alpha_1, \alpha_3, \alpha_4\}$. Therefore E^4 is a quotient of the smaller induced representation $\mathscr{H} \otimes_{\mathscr{H}_I} U$. It follows that V is a quotient of a representation induced from the $A_1 \times E_6$ parabolic in $E_8(k)$. Since $e_{\alpha_1}(s_2s_3\tau') = q^{-1}$, the inducing representation is Steinberg on the A_1 factor. Since $e_{\alpha_3}(s_2s_3\tau') = e_{\alpha_4}(s_2s_3\tau') = q^{-4}$, the E_6-factor is the unique square integrable representation of simply connected $E_6(k)$ containing σ. (See [R4, Section 11], where there are three such representations of adjoint $E_6(k)$, differing by unramified twists. These become mutually isomorphic on the isogenous image of simply connected $E_6(k)$.) A similar analysis can be made with the weight $s_4s_3s_2\tau$ to see that V is a quotient of an induced representation from the $A_2 \times D_5$ parabolic in $E_8(k)$.

Next, we consider the restriction of V to maximal compact subgroups of $E_8(k)$. This is equivalent to restricting E^4 to maximal parahoric subalgebras of $\mathscr{H}^4$ [R4, Section 4]. Since $\tau = \tau_h$, it suffices to restrict to the subalgebra $\mathscr{H}_0$ (the other restrictions then being obtained by restricting to reflection subgroups of $W(F_4)$; see [R5, 5.7]).

There are two methods. First, we invoke our description of E^4 as part of a one-parameter family of modules. Since the operators T_s on E^c have continuous matrix entries for $c > 0$, we can let $c \to 1$ without changing the restriction to the finite-dimensional semisimple subalgebra $\mathscr{H}_0$. Fortunately, the representation E^1 comes from a square-integrable representation of $F_4(k)$. (It can happen that E^4 is square-integrable, but E^1 is not even tempered.) Since E^1 is tempered, we can calculate the restriction in E^1 using results of Lusztig, along with Shoji's calculation of Green polynomials (see [R4, Section 8]). We find that

$$E^4|_{\mathscr{H}_0} = \phi_{(12,4)} + \phi'_{(8,9)} + \phi''_{(8,9)} + \phi_{(9,10)} + \phi_{(4,13)} + \phi_{(1,24)}. \tag{7.1}$$

These are representations of the Weyl group $W(F_4)$, as in [C, 13.2], which correspond to unipotent representations of $E_8(\mathbb{F}_q)$, as tabulated in [C, 13.9].

In the second method, we use the weights to arrive at Eq. (7.1) in another way, which is more elementary and does not rely on a deformation $c \to 1$. Instead, we deform $q \to 1$. The resulting representation $E_{q=1}$ is now a (reducible) representation of the affine Weyl group $\widetilde{W}(F_4)$. We want to determine its restriction to the finite Weyl group $W(F_4)$. From our previous observations on parabolic induction we see that $E_{q=1}$ is contained in both of the representations

$$\mathrm{Ind}^{W(F_4)}_{\langle s_1, s_3, s_4\rangle}[11] \otimes [111], \qquad \mathrm{Ind}^{W(F_4)}_{\langle s_1, s_2, s_4\rangle}[111] \otimes [11],$$

where $[1^n]$ is the sign character of S_n. Decomposing these using Alvis' tables [A], we find that

$$E_{q=1} = a\phi_{(12,4)} + b(\phi'_{(8,9)} + \phi''_{(8,9)}) + c\phi_{(9,10)} + d\phi_{(4,13)} + e\phi_{(1,24)} + f\phi'_{6,6} + g\phi_{16,5},$$

with $a, b, d, e, f, g \leq 1$, $c \leq 2$. Now, from the weight multiplicities, we get

$$\dim E_{q=1} = 42,$$

and one easily calculates (see [R4, (9.5a)]) the trace of s_1 to be

$$\mathrm{tr}(s_1, E_{q=1}) = -10.$$

Using the character table of $W(F_4)$ or [C, 11.3.6], we again obtain the decomposition (7.1).

REFERENCES

[A] D. Alvis, Induce/restrict matrices for Weyl groups tables available on the internet.

[BS] M. Beynon and N. Spaltentstein, Tables of Green polynomials for exceptional groups, Warwick Computer Science Centre Rep. No. 23, 1986.

[B] A. Borel, Admissible representations of a semisimple group over a local field with vectors fixed under an Iwahori subgroup, *Invent. Math.* **35** (1976), 133–159.

[B2] A. Borel, "Linear Algebraic Groups," Springer-Verlag, New York/Berlin, 1991.

[BGG] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, Schubert cells and cohomology of the spaces G/P, *Russian Math. Surveys* **28** (1973), 1–26.

[BK] C. Bushnell and P. Kutzko, "The Admissible Dual of $GL(N)$ via Compact Open Subgroups," Princeton Univ. Press, Princeton, NJ, 1993.

[C] R. Carter, "Finite Groups of Lie Type: Conjugacy Classes and Characters," Wiley, New York, 1985.

[CG] N. Chriss and V. Ginzburg, "Representation Theory and Complex Geometry," Birkhauser, Basel, 1997.

[HM] R. Howe and A. Moy, Hecke algebra isomorphisms for GL_n over a p-adic field *J. Algebra* **131** (1990), 388–424.

[IM] N. Iwahori and H. Matsumoto, On some Bruhat decompositions and the structure of the Hecke ring of the p-adic groups, *Publ. Inst. Hautes Etudes Sci.* **25** (1965), 5–48.

[J] J. C. Jantzen, "Moduln mit einem höchsten Gewicht," Springer Lecture Notes in Mathematics, Vol. 750, Springer-Verlag, New York/Berlin, 1979.

[KL] D. Kazhdan and G. Lusztig, Proof of the Deligne–Langlands conjecture for Hecke algebras *Invent. Math.* **87** (1987), 153–215.

[K] S.-I. Kato, Irreducibility of principal series representations for Hecke algebras of affine type, *J. Fac. Sci. Univ. Tokyo* **28** (1982), 929–943.

[Ki] J.-L. Kim, Hecke algebras of classical groups over p-adic fields and supercuspidal representations, preprint, *Amer. J. Math.* **121** (1999), 967–1029.

[L1] G. Lusztig, Classification of unipotent representations of simple p-adic groups, *Internat. Math. Rse. Notices* **11** (1995), 517–589.

[L2] G. Lusztig, Affine Hecke algebras and their graded version, *J. Amer. Math. Soc.* **2** (1989), 599–635.

[L3] G. Lusztig, Cuspidal local systems and graded Hecke algebras, II, *in* "Representations of Groups," Conference Proceedings of the Canadian Mathematical Society, Vol. 16, pp. 217–275, Canadian Math. Society, Providence, R.I., 1995.

[Mat] H. Matsumoto, "Analyse harmonique dans les systems de Tits bornologiques de type affine," Lecture Notes in Mathematics, Vol. 590, Springer-Verlag, New York/Berlin, 1977.

[M] L. Morris, Tamely ramified intertwining algebras, *Invent. Math.* **114** (1993), 233–274.

[R1] M. Reeder, Nonstandard intertwining operators and the structure of unramified principal series representations of p-adic groups, *Forum. Math.* **9** (1997), 457–516.

[R2] M. Reeder, Whittaker functions, prehomogeneous vector spaces, and standard representations of, *J. Reine. Angew. Math.* **450** (1994), 83–121.

[R3] M. Reeder, Hecke algebras and harmonic analysis on p-adic groups *Amer. J. Math.* **119** (1997), 225–249.

[R4] M. Reeder, Formal degrees and L-packets of unipotent discrete series representations of exceptional p-adic groups, *J. Reine Angew. Math.*, **520** (2000), 37–93.

[R5] M. Reeder, Euler–Poincaré pairings and elliptic representations of p-adic groups and Weyl groups, *Compositio Math.*, to appear.

[Ro] A. Roche, Types and Hecke algebras for principal series representations of split reductive p-adic groups, *Ann. Ecol. Norm. Super.* **31** (1998), 361–413.

[Sho] T. Shoji, Green polynomials for Chevalley groups of type F4, *Comm. Algebra* **10** (1982), 505–543.